Asymptotic stability of stationary solutions to the Euler-Poisson equations for a multicomponent plasma (Mathematical Analysis of Viscous Incompressible Fluid)

鈴木, 政尋

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鈴木政尋

MASAHIRO SUZUKI *

東京工業大学 大学院情報理工学研究科

GRADUATE SCHOOL OF INFORMATION SCIENCE AND ENGINEERING,

TOKYO INSTITUTE OF TECHNOLOGY

1 Introduction

This short paper is concerned with boundary layers of a multicomponent plasma which consists of electrons and several positive ion species. The motion of the multicomponent plasma is governed by the Euler equations for the ion density ρ_i and the ion velocity u_i of the i-th component:

$$(\rho_i)_t + (\rho_i u_i)_x = 0, \tag{1.1a}$$

$$m_i(u_i)_t + \frac{m_i}{2}(u_i^2)_x + \frac{1}{\rho_i}(p_i(\rho_i))_x = e_i\phi_x, \quad i = 1, \ldots, k, \tag{1.1b}$$

coupled with the Poisson equation for the electrostatic potential $-\phi$:

$$\varepsilon_0\phi_{xx} = \sum_{i=1}^{k} e_i\rho_i - e_0\rho_0(\phi). \tag{1.1c}$$

The positive constants m_i and e_i denote the mass and the charge of the i-th ion, respectively. In addition, ε_0 is permittivity. The pressure p_i is assumed to be a function of the electron density ρ_i given by

$$p_i(\rho_i) = \kappa T_i \rho_i,$$

*masahiro@is.titech.ac.jp

where κ is the Boltzmann constant and T_i is the temperature of the i-th ion. We assume that the electron density ρ_0 obeys the Boltzmann relation, that is,

$$\rho_0(\phi) = \rho_{0+} \exp\left(-\frac{e_0\phi}{\kappa T_0}\right),$$

where the positive constants ρ_{0+}, e_0 and T_0 denote the reference density value, the charge and the temperature of the electron, respectively.

We study an initial-boundary value problem to (1.1) over a one-dimensional half space $\mathbb{R}_+ := \{x > 0\}$, where the initial and the boundary data are prescribed as

$$(\rho_i, u_i)(0, x) = (\rho_{i0}, u_{i0})(x), \quad \lim_{x\to\infty} (\rho_{i0}, u_{i0})(x) = (\rho_{i+}, u_{i+}), \quad i = 1, \dots, k, \tag{1.2}$$

$$\phi(t, 0) = \phi_b. \tag{1.3}$$

Here, ρ_{i+}, u_{i+} and ϕ_b are constants. We take a reference point of the potential ϕ at $x = \infty$, that is,

$$\lim_{x\to\infty} \phi(t, x) = 0.$$

In order to solve the Poisson equation (1.1c) in classical sense, the quasi-neutrality condition is required:

$$\sum_{i=0}^{k} e_i\rho_{i+} - e_0\rho_{0+} = 0. \tag{1.4}$$

The solution of this initial-boundary value problem is constructed in the region where the positivity of the density (1.5) and the supersonic outflow condition (1.6) hold, that is,

$$\inf_{x\in\mathbb{R}_+} \rho_i > 0 \quad \text{for} \quad i = 1, \dots k, \tag{1.5}$$

$$\inf_{x\in\mathbb{R}_+} \left(m_i u_i^2 - \kappa T_i\right) > 0, \quad \sup_{x\in\mathbb{R}_+} u_i < 0 \quad \text{for} \quad i = 1, \dots k. \tag{1.6}$$

Therefore we suppose that the initial data satisfies the same conditions:

$$\inf_{x\in\mathbb{R}_+} \rho_{i0} > 0, \quad \inf_{x\in\mathbb{R}_+} \left(m_i u_{i0}^2 - \kappa T_i\right) > 0, \quad \sup_{x\in\mathbb{R}_+} u_{i0} < 0, \tag{1.7}$$

$$\rho_{i+} > 0, \quad m_i u_{i+}^2 - \kappa T_i > 0, \quad u_{i+} < 0. \tag{1.8}$$

Before we close this section, we briefly discuss about the physical background of our problem and the related mathematical works. A boundary layer problem occurs in plasma devices when the plasma contacts with a surface. Due to the difference of the mobilities of electrons and positive ions, the surface has a negative potential with respect to the plasma. The non-neutral potential region between the plasma and the surface is called a sheath. The readers are referred to [4,5,7]. For the sheath formation, physical observation requires that the positive ions must enter the sheath region with a high velocity. Bohm in [3] treated a simple case when the plasma contains electrons and only one component of mono-valence ions and derived the original Bohm criterion for the velocity u_1:

$$\kappa T_0 + \kappa T_1 < m_1 u_{1+}^2, \quad u_{1+} < 0. \tag{1.9}$$

By studying the stationary problem of the system (1.1), Riemann in [8] obtained the generalized Bohm criterion for the multicomponent plasma. This criterion claims that the velocity of positive ions should satisfy (1.8) and

$$B_+ := -\sum_{i=1}^{k} \frac{e_i^2 \rho_{i+}}{m_i u_{i+}^2 - \kappa T_i} + \frac{e_0^2 \rho_{0+}}{\kappa T_0} > 0. \tag{1.10}$$

Let us mention mathematical results which study the sheath formation and the original Bohm criterion (1.9). Ambroso, Méhats and P.-A. Raviart in [2] showed the existence of the monotone stationary solution to (1.1) with $k = 1$ under (1.9) over a one-dimensional bounded domain. Later Ambroso in [1] numerically showed that the solution to (1.1) approaches the stationary solution as time tends to infinity in the same setting as in [2]. Suzuki in [9] interpreted the sheath to be a monotone stationary solution to (1.1) with $k = 1$ over a one-dimensional half space and showed that the Bohm criterion is sufficient for the unique existence of the monotone stationary solution. In [6], the asymptotic stability of the stationary solution is proved under (1.9). Consequently, these results ensure the mathematical validity of the original Bohm criterion (1.9).

In this short paper, we briefly study the rigorous justification of the generalized Bohm criterion (1.10). More precisely, we introduce the existence and the stability theorems on the stationary solution to the system (1.1) for the multicomponent plasma. For the detailed discussion on this research, please see the paper [10].

2 Unique existence of the stationary solution

This section is devoted to the discussion on the unique existence of the monotone stationary solution. The stationary solution $(\tilde{\rho}_1, \tilde{u}_1, \ldots, \tilde{\rho}_k, \tilde{u}_k, \tilde{\phi})$ is a solution to (1.1) independent of the time variable t. Hence, it verifies

$$(\tilde{\rho}_i \tilde{u}_i)_x = 0, \tag{2.1a}$$

$$\frac{m_i}{2}(\tilde{u}_i^2)_x + \frac{\kappa T_i}{\tilde{\rho}_i}(\tilde{\rho}_i)_x = e_i \tilde{\phi}_x, \quad i = 1, \ldots, k, \tag{2.1b}$$

$$\varepsilon_0 \tilde{\phi}_{xx} = \sum_{i=1}^{k} e_i \tilde{\rho}_i - e_0 \rho_{0+} \exp\left(-\frac{e_0 \tilde{\phi}}{\kappa T_0}\right) \tag{2.1c}$$

and the conditions (1.2)–(1.6):

$$\lim_{x\to\infty}(\tilde\rho_1,\tilde u_1,\ldots,\tilde\rho_k,\tilde u_k)(x)=(\rho_{1+},u_{1+},\ldots,\rho_{k+},u_{k+}), \tag{2.2a}$$

$$\tilde\phi(0)=\phi_b,\quad \lim_{x\to\infty}\tilde\phi(x)=0, \tag{2.2b}$$

$$\inf_{x\in\mathbb{R}_+}\tilde\rho_i>0,\quad \inf_{x\in\mathbb{R}_+}\left(m_i\tilde u_i^2-\kappa T_i\right)>0,\quad \sup_{x\in\mathbb{R}_+}\tilde u_i<0,\quad i=1,\ldots,k. \tag{2.2c}$$

The key of the proof of the existence theorem is reduction of the system (2.1) to a scalar equation for $\tilde\phi$. Assuming the existence of the monotone solution satisfying

$$(\tilde\rho_1,\tilde u_1,\ldots,\tilde\rho_k,\tilde u_k)\in C^1(\mathbb{R}_+),\qquad \tilde\phi\in C(\overline{\mathbb{R}_+})\cap C^2(\mathbb{R}_+) \tag{2.3}$$

to the stationary problem (2.1) and (2.2), we drive the scalar equation which $\tilde\phi$ satisfies. Integrating (2.1a) over (x,∞) gives

$$\tilde u_i(x)=\frac{\rho_{i+}u_{i+}}{\tilde\rho_i(x)}\quad\text{for }\ i=1,\ldots,k. \tag{2.4}$$

Substitute (2.4) in (2.1b), divide the result by $\tilde\rho$ and then integrate over (x,∞) to obtain

$$e_i\tilde\phi(x)=f_i(\tilde\rho_i(x))\quad\text{for }\ i=1,\ldots,k, \tag{2.5}$$

where f_i is defined by

$$f_i(\tilde\rho_i):=\kappa T_i\log\tilde\rho_i+m_i\frac{\rho_{i+}^2u_{i+}^2}{2\tilde\rho_i^2}-\kappa T_i\log\rho_{i+}-m_i\frac{u_{i+}^2}{2}.$$

By restricting the domain $D(f_i)$ into $(0,\rho_{i+}M_{i+}]$, where $M_{i+}^2=m_iu_{i+}^2/\kappa T_i$, we see that f_i is invertible. Then it holds that

$$\tilde\rho_i(x)=f_i^{-1}(e_i\tilde\phi(x))\quad\text{for }\ i=1,\ldots,k. \tag{2.6}$$

Substitute (2.6) in (2.1c), multiply the resultant equation by $\tilde\phi_x$, integrate the result over (x,∞) and then use the condition (2.2a) and $\lim_{x\to\infty}\tilde\phi_x(x)=0$ to obtain the scalar equation for ϕ:

$$\frac{\varepsilon_0}{2}(\tilde\phi_x)^2=V(\tilde\phi),\quad V(\tilde\phi):=\int_0^{\tilde\phi}\sum_{i=1}^k e_if_i^{-1}(e_i\eta)-e_0\rho_{0+}\exp\left(-\frac{e_0\eta}{\kappa T_0}\right)d\eta, \tag{2.7}$$

where V is called as the Sagdeev potential in plasma physics. This equation requires the necessary condition $V(\phi_b)\geq 0$.

On the other hand, if the problem (2.7) and (2.2b) has a monotone solution $\tilde\phi\in C(\overline{\mathbb{R}_+})\cap C^2(\mathbb{R}_+)$, then we can easily check that

$$(\tilde\rho_1,\tilde u_1,\ldots,\tilde\rho_k,\tilde u_k,\tilde\phi):=\left(f_1^{-1}(e_1\tilde\phi),\frac{\rho_{1+}u_{1+}}{f_1^{-1}(e_1\tilde\phi)},\ldots,f_k^{-1}(e_k\tilde\phi),\frac{\rho_{k+}u_{k+}}{f_k^{-1}(e_k\tilde\phi)},\tilde\phi\right)$$

is a monotone stationary solution to (2.1) and (2.2). The uniqueness of the monotone stationary solution to (2.1) and (2.2) also follows from the uniqueness of the solution to (2.7) and (2.2b).

Hence, it is sufficient to show the unique solvability of the problem (2.7) and (2.2b). We can solve this problem by virtue of the standard ODE theory. The unique existence theorem is as follows.

Theorem 2.1. *Let the asymptotic state* $(\rho_{1+}, u_{1+}, \ldots, \rho_{k+}, u_{k+})$ *satisfy* (1.4) *and* (1.8).

(*i*) *Suppose* $B_+ > 0$. *Then there exists a certain positive constant* δ *such that if* $|\phi_b| \leq \delta$, *the stationary problem* (2.1) *and* (2.2) *has a unique monotone stationary solution* $(\tilde{\rho}_1, \tilde{u}_1, \ldots, \tilde{\rho}_k, \tilde{u}_k, \tilde{\phi})$ *verifying* (2.3).

(*ii*) *Suppose* $B_+ = 0$. *Then there exists a certain positive constant* δ *such that if* $|\phi_b| \leq \delta$ *and* $V(\phi_b) \geq 0$, *the stationary problem* (2.1) *and* (2.2) *has a unique monotone stationary solution* $(\tilde{\rho}_1, \tilde{u}_1, \ldots, \tilde{\rho}_k, \tilde{u}_k, \tilde{\phi})$ *verifying* (2.3).

(*iii*) *Suppose* $B_+ < 0$. *If* $\phi_b \neq 0$, *no stationary solution verifying* (2.3) *exists. If* $\phi_b = 0$, *a constant sate* $(\tilde{\rho}_1, \tilde{u}_1, \ldots, \tilde{\rho}_k, \tilde{u}_k, \tilde{\phi}) = (\rho_{1+}, u_{1+}, \ldots, \rho_{k+}, u_{k+}, 0)$ *is the unique stationary solution.*

The author in [9] constructed non-monotone stationary solutions for the case $k = 1$. Thus the monotonicity is necessary to show the uniqueness.

3 Asymptotic stability of the stationary solution

Before stating our stability theorem, let us mention difficulties of our stability analysis. For notational convenience, we introduce the perturbation from the asymptotic state $(\rho_{1+}, u_{1+}, \ldots, \rho_{k+}, u_{k+}, 0)$ as

$$\psi_i := \rho_i - \rho_{i+}, \quad \eta_i := u_i - u_{i+}, \quad i = 1, \ldots, k,$$
$$\sigma := \phi - \tilde{\phi}.$$

Linearizing the system (1.1) around the asymptotic state gives

$$\psi_{it} + u_{i+}\psi_{ix} + \rho_{i+}\eta_{ix} = 0, \tag{3.1a}$$

$$\eta_{it} + u_{i+}\eta_{ix} + \frac{\kappa T_0}{m_i \rho_{i+}}\psi_{ix} = \frac{e_i}{m_i}\sigma_x, \quad i = 1, \ldots, k, \tag{3.1b}$$

$$\varepsilon_0 \sigma_{xx} - \frac{e_0^2 \rho_{0+}}{\kappa T_0}\sigma = \sum_{i=1}^{k} e_i \psi_i. \tag{3.1c}$$

Notice that the real part of all spectra of this system is zero under the assumption

$$u_+ = u_{1+} = \cdots = u_{k+}. \tag{3.2}$$

This causes our problem to be difficult since standard methods are not applicable. For overcoming this issue, we employ the weighted Sobolve space with a weight function

$$(1+\beta x)^{\lambda} \quad \text{or} \quad e^{\beta x}.$$

We briefly discuss about effectiveness of the weighted Sobolve space in our analysis under the criterion (1.10). Multiply (3.1) by $e^{\beta x/2}$ and introduce new unknown function $P_i := e^{\beta x/2}\psi_i$, $Q_i := e^{\beta x/2}\eta_i$ and $R := e^{\beta x/2}\sigma$. Moreover, rewriting the system (3.1) for P_i, Q_i and R gives

$$P_{it} + u_{i+}P_{ix} + \rho_{i+}Q_{ix} - \frac{\beta}{2}(u_{i+}P_i + \rho_{i+}Q_i) = 0, \tag{3.3a}$$

$$Q_{it} + u_{i+}Q_{ix} + \frac{\kappa T_0}{m_i\rho_{i+}}P_{ix} - \frac{\beta}{2}\left(u_{i+}Q_i + \frac{\kappa T_0}{m_i\rho_{i+}}P_i\right) = \frac{e_i}{m_i}R_x - \frac{\beta e_i}{2m_i}R, \quad i = 1, \ldots, k, \tag{3.3b}$$

$$\varepsilon_0\left(Q_{xx} - \beta Q_x + \frac{\beta^2}{4}Q\right) - \frac{e_0^2\rho_{0+}}{\kappa T_0}Q = \sum_{i=1}^{k} e_i P_i. \tag{3.3c}$$

By applying spectral analysis to the system (3.3), we have

Proposition 3.1. *Let the asymptotic state* $(\rho_{1+}, u_{1+}, \ldots, \rho_{k+}, u_{k+})$ *satisfy* (1.8) *and* (3.2). *Then the following two conditions are equivalent:*

(*i*) *The real part of all spectra of* (3.3) *in the whole space* $\mathbb{R}$ *is negative for sufficiently small* $\beta > 0$.

(*ii*) *The generalized Bohm criterion* (1.10) *holds.*

Although our problem is the boundary value problem, Proposition 3.1 implies that the weighted Sobolve space is useful in our stability analysis. The stability theorem is summarized in Theorem 3.2. The proof is based on the combination of the weighted energy method and Fourier analysis.

Theorem 3.2. *Let the asymptotic state* $(\rho_{1+}, u_{1+}, \ldots, \rho_{k+}, u_{k+})$ *satisfy the conditions* (1.4), (1.8), (1.10) *and* $u_{1+} = \cdots = u_{k+}$.

(*i*) *Suppose that* $e^{\alpha x/2}(\rho_{i0} - \tilde{\rho}_i)$ *and* $e^{\alpha x/2}(u_{i0} - \tilde{u}_i)$ *belong to the Sobolve space* $H^2(\mathbb{R}_+)$ *for* $i = 1, \ldots, k$, *where* α *is some positive constant. Then there exist positive constants* $\beta(\leq \alpha)$ *and* δ *such that if*

$$|\phi_b| + \sum_{i=1}^{k} \|(e^{\beta x/2}(\rho_{i0} - \tilde{\rho}_i), e^{\beta x/2}(u_{i0} - \tilde{u}_i))\|_{H^2} \leq \delta,$$

the initial-boundary value problem (1.1)–(1.3) *has a unique solution* $(\rho_1, u_1, \ldots, \rho_k, u_k, \phi)$ *satisfying*

$$\left(e^{\beta x/2}(\rho_i - \tilde{\rho}_i), e^{\beta x/2}(u_i - \tilde{u}_i)\right) \in \bigcap_{j=0}^{2} C^j([0,\infty); H^{2-j}) \quad \textit{for} \quad i = 1, \ldots, k,$$
$$e^{\beta x/2}(\phi - \tilde{\phi}) \in \bigcap_{j=0}^{2} C^j([0,\infty); H^{4-j}).$$

Moreover it verifies the decay estimate

$$\sup_{x\in\mathbb{R}_+} |(\rho_1 - \tilde{\rho}_1, u_1 - \tilde{u}_1, \ldots, \rho_k - \tilde{\rho}_k, u_k - \tilde{u}_k, \phi - \tilde{\phi})(t)| \le Ce^{-\gamma t},$$

where positive constants C *and* γ *are independent of the time variable* t.

(*ii*) *Let* λ *and* ν *satisfy* $\lambda \ge 2$ *and* $\nu \in (0, \lambda]$. *Suppose that* $(1+\alpha x)^{\lambda/2}(\rho_{i0} - \tilde{\rho}_i)$ *and* $(1+\alpha x)^{\lambda/2}(u_{i0} - \tilde{u}_i)$ *belong to the Sobolve space* $H^2(\mathbb{R}_+)$ *for* $i = 1, \ldots, k$, *where* α *is some positive constant. Then there exist positive constants* $\beta(\le \alpha)$ *and* δ *such that if*

$$|\phi_b| + \sum_{i=1}^{k} \|((1+\alpha x)^{\lambda/2}(\rho_{i0} - \tilde{\rho}_i), (1+\alpha x)^{\lambda/2}(u_{i0} - \tilde{u}_i))\|_{H^2} \le \delta,$$

the initial-boundary value problem (1.1)–(1.3) *has a unique solution* $(\rho_1, u_1, \ldots, \rho_k, u_k, \phi)$ *satisfying*

$$\left((1+\alpha x)^{\lambda/2}(\rho_i - \tilde{\rho}_i), (1+\alpha x)^{\lambda/2}(u_i - \tilde{u}_i)\right) \in \bigcap_{j=0}^{2} C^j([0,\infty); H^{2-j}) \quad \textit{for} \quad i = 1, \ldots, k,$$
$$(1+\alpha x)^{\lambda/2}(\phi - \tilde{\phi}) \in \bigcap_{j=0}^{2} C^j([0,\infty); H^{4-j}).$$

Moreover it verifies the decay estimate

$$\sup_{x\in\mathbb{R}_+} |(\rho_1 - \tilde{\rho}_1, u_1 - \tilde{u}_1, \ldots, \rho_k - \tilde{\rho}_k, u_k - \tilde{u}_k, \phi - \tilde{\phi})(t)| \le C(1+\beta t)^{-\lambda+\zeta}$$

for an arbitrary $\zeta \in [\nu, \lambda]$, *where the positive constant* C *is independent of the time variable* t.

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